Pacific tides meet along a line of comparatively still water, on which the ice accumulates so as to form an impassable barrier. This line passes through the following points:—North of Renslœr Harbour; North of Wellington Channel; Banks's Strait; North of Prince of Wales's Strait.

A general sketch of this theory was published, in April, 1858, in the "Natural History Review," and communicated by Mr. Haughton to Captain M'Clintock in the summer of 1857. The practical conclusion deduced from this theory was, that no ship could pass from the Atlantic to the Pacific, or *vice versâ*. Mr. Haughton stated that the discoveries of Captain M'Clintock respecting the "Erebus" and "Terror" confirmed this theory.

The Rev. Robert Carmichael read the first part of a paper—

ON CERTAIN METHODS IN THE CALCULUS OF FINITE DIFFERENCES.

Sect. I.—*On the Solution of Systems of Simultaneous Equations in the Calculus of Finite Differences.*

There are many questions in physics, more especially in the departments of magnetism, heat, meteorology, and astronomy, in which the conditions for solution are given by observations made at periods separated by finite intervals, and the unknown quantities enter in certain linear combinations. It would seem to be obvious that, in some of these cases, the phenomena would admit, as their possible analytical expression, and in some cases would require as their only suitable expression, systems of simultaneous equations in finite differences.

For instance, it seems evident that there are certain classes of physical problems which are only susceptible of exposition by such a system as,—if u_t, v_t, w_t, &c., are unknown functions of t to be determined, a_1, b_1, c_1, &c., constants, and f_1, f_2, f_3, &c., functions of known form,—

$$\left.\begin{array}{l} \Delta . u_t = a_1 u_t + b_1 v_t + c_1 w_t + \ldots + f_1(t) \\ \Delta . v_t = a_2 u_t + b_2 v_t + c_2 w_t + \ldots + f_2(t) \\ \Delta . w_t = a_3 u_t + b_3 v_t + c_3 w_t + \ldots + f_3(t) \\ \qquad\qquad \&c. \end{array}\right\},$$

or,

$$\left.\begin{array}{l} u_{t+n} = a_1 u_t + b_1 v_t + c_1 w_t + \ldots + f_1(t) \\ v_{t+n} = a_2 u_t + b_2 v_t + c_2 w_t + \ldots + f_2(t) \\ w_{t+n} = a_3 u_t + b_3 v_t + c_3 w_t + \ldots + f_3(t) \\ \qquad\qquad \&c. \end{array}\right\},$$

So far as I am aware, no general method has been given for the solution of such systems of equations. In the following pages an attempt is made to supply the desideratum.

1. Let it be proposed to solve the system of equations in finite differences, of the first order, exhibiting n unknown functions,

$$\left.\begin{array}{l} u_{x+1} = a_1 u_x + b_1 v_x + c_1 w_x + \ldots \\ v_{x+1} = a_2 u_x + b_2 v_x + c_2 w_x + \ldots \\ w_{x+1} = a_3 u_x + b_3 v_x + c_3 w_x + \ldots \\ \qquad\qquad \text{\&c.} \end{array}\right\},$$

the auxiliary known functions f_1, f_2, f_3, &c., being, for the present, omitted.

Multiply the first equation by λ, the second by μ, the third by ν, &c.; then, adding all together, we get

$$e^{D_x}(\lambda u_x + \mu v_x + \nu w_x + \ldots) = \left\{\begin{array}{c} (a_1\lambda + a_2\mu + a_3\nu + \ldots)\, u_x \\ + \\ (b_1\lambda + b_2\mu + b_3\nu + \ldots)\, v_x \\ + \\ (c_1\lambda + c_2\mu + c_3\nu + \ldots) w_x \\ + \text{\&c.} \end{array}\right.$$

Now, as we have introduced n arbitrary constants, we are at liberty to subject them to n conditions, which we may suppose to be—

$$\begin{array}{l} a_1\lambda + a_2\mu + a_3\nu + \ldots = k\lambda, \\ b_1\lambda + b_2\mu + b_3\nu + \ldots = k\mu, \\ c_1\lambda + c_2\mu + c_3\nu + \ldots = k\nu, \\ \qquad\qquad \text{\&c.,} \end{array}$$

k being a new constant.

The preceding equation is thus reduced to the form

$$e^{D_x}(\lambda u_x + \mu v_x + \nu w_x + \ldots) = k(\lambda u_x + \mu v_x + \nu w_x + \ldots),$$

the solution of which is, at once,

$$\lambda u_x + \mu v_x + \nu w_x + \ldots = Ck^x,$$

where C is any arbitrary constant.

Now, with regard to the quantity k, it is to be observed that if $(n-1)$ of the quantities λ, μ, ν, &c., be eliminated between the assumed equations of connexion, the n^{th} quantity will of course disappear of itself, and we obtain an equation of the n^{th} degree in k, and the known quantities a_1, b_1, c_1, &c., namely, the determinant,

$$\left|\begin{array}{cccc} a_1 - k, & a_2\ , & a_3\ , & \text{\&c.} \\ b_1\ , & b_2 - k, & b_3\ , & \text{\&c.} \\ c_1\ , & c_2\ , & c_3 - k, & \text{\&c.} \\ & \text{\&c., \&c.} & & \end{array}\right| = 0:$$

consequently, in the above solution, k may be supposed to have any one of n known values.

Hence, writing down the series of solutions corresponding to the several roots k_1, k_2, k_3, &c., it is obvious that the general solution of the given system of simultaneous equations in finite differences is exponible in the form—

$$\left.\begin{array}{l} u_x = C_1k_1^x + C_2k_2^x + C_3k_3^x + \ldots + C_nk_n^x \\ v_x = D_1k_1^x + D_2k_2^x + D_3k_3^x + \ldots + D_nk_n^x \\ w_x = E_1k_1^x + E_2k_2^x + E_3k_3^x + \ldots + E_nk_n^x \\ \qquad\qquad\qquad \text{\&c.} \end{array}\right\},$$

where, of the constants C_1, D_1, E_1, &c., n only are arbitrary.

When some of the roots k_1, k_2, k_3, &c., are equal, or when there are pairs of imaginary roots, modifications sufficiently evident must be introduced in the general form of solution.

Thus, in the case of r equal roots, whose common value is k_1, the general form of solution becomes

$$\left.\begin{array}{l} u_x = k_1^x(C_{r-1}x^{r-1} + C_{r-2}x^{r-2} + \ldots + C_1x + C_0) + \ldots + C_nk_n^x \\ v_x = k_1^x(D_{r-1}x^{r-1} + D_{r-2}x^{r-2} + \ldots + D_1x + D_0) + \ldots + D_nk_n^x \\ w_x = k_1^x(E_{r-1}x^{r-1} + E_{r-2}x^{r-2} + \ldots + E_1x + E_0) + \ldots + E_nk_n^x \\ \qquad\qquad\qquad \text{\&c.} \end{array}\right\},$$

Lastly, in the case of a pair of imaginary roots, the general form of solution becomes—

$$\left.\begin{array}{l} u_x = C_1(k_1 + k_2\sqrt{-1})^x + C_2(k_1 - k_2\sqrt{-1})^x + C_3k_3^x + \ldots + C_nk_n^x \\ v_x = D_1(k_1 + k_2\sqrt{-1})^x + D_2(k_1 - k_2\sqrt{-1})^x + D_3k_3^x + \ldots + D_nk_n^x \\ w_x = E_1(k_1 + k_2\sqrt{-1})^x + E_2(k_1 - k_2\sqrt{-1})^x + E_3k_3^x + \ldots + E_nk_n \\ \qquad\qquad\qquad \text{\&c.} \end{array}\right\},$$

which may obviously be reduced to the simpler form—

$$\left.\begin{array}{l} u_x = (k_1^2 + k_2^2)^{\frac{x}{2}}\, C'_1 \cos\left\{x \tan^{-1}\left(\frac{k_2}{k_1}\right) + C'_2\right\} + C_3k_3^x + \ldots + C_nk_n^x \\ v_x = (k_1^2 + k_2^2)^{\frac{x}{2}}\, D'_1 \cos\left\{x \tan^{-1}\left(\frac{k_2}{k_1}\right) + D'_2\right\} + D_3k_3^x + \ldots + D_nk_n^x \\ w_x = (k_1^2 + k_2^2)^{\frac{x}{2}}\, E'_1 \cos\left\{x \tan^{-1}\left(\frac{k_2}{k_1}\right) + E'_2\right\} + E_3k_3^x + \ldots + E_nk_n^x \\ \qquad\qquad\qquad \text{\&c.} \end{array}\right\},$$

EXAMPLES.

(1.) Let it be proposed to solve the system of two simultaneous equations—

$$\left.\begin{array}{l} u_{x+1} = a_1 u_x + b_1 v_x \\ v_{x+1} = a_2 u_x + b_2 v_x \end{array}\right\}.$$

The solution is at once—

$$u_x = C_1 k_1^x + C_2 k_2^x,$$

$$v_x = C_1 \left(\frac{k_1 - a_1}{b_1}\right) k_1^x + C_2 \left(\frac{k_2 - a_1}{b_1}\right) k_2^x,$$

where k_1, k_2 are the roots of the equation

$$(k - a_1)(k - b_2) = b_1 a_2.$$

(2.) Let it be proposed to solve the system of three simultaneous equations—

$$\left.\begin{array}{l} u_{x+1} = b_1 v_x + c_1 w_x \\ v_{x+1} = a_2 u_x + c_2 w_x \\ w_{x+1} = a_3 u_x + b_3 v_x \end{array}\right\},$$

The solution is at once

$$\left.\begin{array}{l} u_x = C_1 k_1^x + C_2 k_2^x + C_3 k_3^x \\ v_x = D_1 k_1^x + D_2 k_2^x + D_3 k_3^x \\ w_x = E_1 k_1^x + E_2 k_2^x + E_3 k_3^x \end{array}\right\},$$

where

$$D_1 = C_1 \frac{a_2 k_1 + c_2 a_3}{k_1^2 - c_2 b_3}, \qquad E_1 = C_1 \frac{a_3 k_1 + b_3 a_2}{k_1^2 - c_2 b_3},$$

$$D_2 = C_2 \frac{a_2 k_2 + c_2 a_3}{k_2^2 - c_2 b_3}, \qquad E_2 = C_2 \frac{a_3 k_2 + b_3 a_2}{k_2^2 - c_2 b_3},$$

$$D_3 = C_3 \frac{a_2 k_3 + c_2 a_3}{k_3^2 - c_2 b_3}, \qquad E_3 = C_3 \frac{a_3 k_3 + b_3 a_2}{k_3^2 - c_2 b_3},$$

and where k_1, k_2, k_3 are the roots of the cubic equation obtained by the elimination of λ, μ, ν between the equations

$$\left.\begin{array}{l} a_2\mu + a_3\nu = k\lambda \\ b_1\lambda + b_3\nu = k\mu \\ c_1\lambda + c_2\mu = k\nu \end{array}\right\}, \quad \text{or} \quad \begin{vmatrix} -k, & a_2, & a_3 \\ b_1, & -k, & b_3 \\ c_1, & c_2, & -k \end{vmatrix} = 0,$$

or,

$$k^3 - (b_1 a_2 + c_1 a_3 + c_2 b_3) k - (c_2 b_1 a_3 + c_1 a_2 b_3) = 0.$$

As regards the determination of the arbitrary constants C_1, C_2, C_3, we are supposed to be given the values of u_x, v_x, w_x, corresponding to a particular value of x. Let it be supposed, for example, that the values of

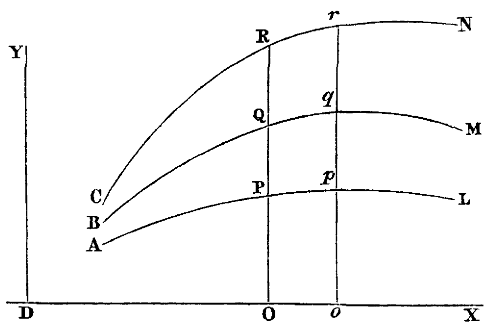
these quantities, corresponding to $x=0$, are respectively, α, β, γ; then we shall have, for the determination required,

$$\left.\begin{aligned}\alpha &= C_1 + C_2 + C_3\\ \beta &= D_1 + D_2 + D_3\\ \gamma &= E_1 + E_2 + E_3\end{aligned}\right\},$$

writing, for simplicity, the constants D_1, E_1, &c., in their primitive form.

The following is the geometrical problem, of which the given system of simultaneous equations of finite differences is the analytical statement.

To find *three* curves AL, BM, CN, such that,



taking $OD = x$ and Oo as the unit-increment of the abscissa,

$$\begin{aligned} OP &= u_x, & & & op &= u_{x+1},\\ OQ &= v_x, & &\text{whence} & oq &= v_{x+1},\\ OR &= w_x, & & & or &= w_{x+1},\end{aligned}$$

we may have

$$\left.\begin{aligned} op &= b_1 \,.\, OQ + c_1 \,.\, OR\\ oq &= a_2 \,.\, OP + c_2 \,.\, OR\\ or &= a_3 \,.\, OP + b_3 \,.\, OQ\end{aligned}\right\},$$

2. In the preceding article, the solution of the system

$$\left.\begin{aligned} u_{x+1} &= a_1u_x + b_1v_x + c_1w_x + \;.\;.\;.\\ v_{x+1} &= a_2u_x + b_2v_x + c_2w_x + \;.\;.\;.\\ w_{x+1} &= a_3u_x + b_3v_x + c_3w_x + \;.\;.\;.\\ &\quad\&\text{c.}\end{aligned}\right\},$$

was found by the aid of indeterminate constants. The following is, perhaps, a simpler and more elegant method.

Writing these equations in the form

$$\left.\begin{array}{c}(a_1 - e^D)\,u_x + b_1 v_x + c_1 w_x + \ldots = 0 \\ a_2 u_x + (b_2 - e^D)\,v_x + c_2 w_x + \ldots = 0 \\ a_3 u_x + b_3 v_x + (c_3 - e^D)\,w_x + \ldots = 0 \\ \&\text{c.}\end{array}\right\},$$

and remembering that the symbol e^D is commutative with constants, we get as the result for u_x, the determinant

$$\left\{\begin{array}{cccc}a_1 - e^D, & b_1, & c_1, & \&\text{c.} \\ a_2, & b_2 - e^D, & c_2, & \&\text{c.} \\ a_3, & b_3, & c_3 - e^D, & \&\text{c.} \\ & \&\text{c.}, \&\text{c.} & & \end{array}\right\} . \, u_x = 0,$$

the first term of which is, of course,

$$(a_1 - e^D)\,(b_2 - e^D)\,(c_3 - e^D)\,\ldots\,u_x.$$

Consequently, the result is an equation of the form

$$u_{x+n} + \alpha u_{x+n-1} + \beta u_{x+n-2} + \&\text{c.} + \sigma u_x = 0,$$

where α, β, γ, &c., σ, are constants; and if the roots of the correspondent symbolic equation be, as before, k_1, k_2, k_3, &c., the value of u_x is, as stated,

$$u_x = C_1 k_1^{\,x} + C_2 k_2^{\,x} + C_3 k_3^{\,x} + \&\text{c.}$$

3. If the system of equations to be solved had been

$$\left.\begin{array}{l}\Delta\,.\,u_x = a_1 u_x + b_1 v_x + c_1 w_x + \ldots \\ \Delta\,.\,v_x = a_2 u_x + b_2 v_x + c_2 w_x + \ldots \\ \Delta\,.\,w_x = a_3 u_x + b_3 v_x + c_3 w_x + \ldots \\ \qquad\qquad \&\text{c.}\end{array}\right\},$$

it is evident that the solution required will be had by simply substituting in the results of the previous article $(a_1 + 1)$, $(b_2 + 1)$, $(c_3 + 1)$, &c., for a_1, b_2, c_3, &c., respectively.

EXAMPLES.

(1.) Thus, if it be required to solve the system of two equations

$$\left.\begin{array}{l}\Delta\,.\,u_x = a_1 u_x + b_1 v_x \\ \Delta\,.\,v_x = a_2 u_x + b_2 v_x\end{array}\right\},$$

we have, at once,

$$u_x = C_1 m_1^{\,x} + C_2 m_2^{\,x},$$

$$v_x = C_1 \left(\frac{m_1 - a_1 - 1}{b_1}\right) m_1^{\,x} + C_2 \left(\frac{m_2 - a_1 - 1}{b_1}\right) m_2^{\,x},$$

where m_1, m_2 are the roots of the equation

$$(m-a_1-1)\,(m-b_2-1)=b_1a_2.$$

(2.) Again, if it be required to solve the system of three simultaneous equations

$$\left.\begin{aligned}\Delta\,.\,u_x &= b_1v_x+c_1w_x\\ \Delta\,.\,v_x &= a_2u_x+c_2w_x\\ \Delta\,.\,w_x &= a_3u_x+b_3v_x\end{aligned}\right\},$$

we have for the result sought

$$\left.\begin{aligned}u_x &= C_1m_1^{\,x}+C_2m_2^{\,x}+C_3m_3^{\,x}\\ v_x &= D_1m_1^{\,x}+D_2m_2^{\,x}+D_3m_3^{\,x}\\ w_x &= E_1m_1^{\,x}+E_2m_2^{\,x}+E_3m_3^{\,x}\end{aligned}\right\},$$

where m_1, m_2, m_3 are the roots of the cubic equation obtained by the elimination of λ, μ, ν, between the equations

$$\left.\begin{aligned}a_2\mu+a_3\nu &= (m-1)\,\lambda\\ b_1\lambda+b_3\nu &= (m-1)\,\mu\\ c_1\lambda+c_2\mu &= (m-1)\,\nu\end{aligned}\right\},$$

or,

$$(m-1)^3-(b_1a_2+c_1a_3+c_2b_3)\,(m-1)-(c_2b_1a_3+c_1a_2b_3)=0.$$

It will be observed that the form in which, in the correlative example, D_1, E_1, &c., are expressed in terms of C_1, C_2, C_3, remains unaffected, as it should.

4. It is plain that we may employ a method similar to that just given, for the solution of the system of simultaneous equations in finite differences of the n^{th} order,

$$\left.\begin{aligned}u_{x+n} &= a_1u_x+b_1v_x+c_1w_x+\ldots\\ v_{x+n} &= a_2u_x+b_2v_x+c_2w_x+\ldots\\ w_{x+n} &= a_3u_x+b_3v_x+c_3w_x+\ldots\\ &\text{\&c.}\end{aligned}\right\},$$

The reduct equation is, in this case,

$$e^{nD}\,.\,(\lambda u_x+\mu v_x+\nu w_x+\ldots)=k^n\,(\lambda u_x+\mu v_x+\nu w_x+\ldots),$$

the equations of condition being

$$\left.\begin{aligned}a_1\lambda+a_2\mu+a_3\nu+\ldots &= k^n\lambda\\ b_1\lambda+b_2\mu+b_3\nu+\ldots &= k^n\mu\\ c_1\lambda+c_2\mu+c_3\nu+\ldots &= k^n\nu\\ &\text{\&c.}\end{aligned}\right\},$$

and the solution of the reduct equation is, if α, α', α'', &c., be the n several roots of unity,

$$\lambda u_x + \mu v_x + \nu w_x + \ldots = C(\alpha k)^x + C'(\alpha' k)^x + C''(\alpha'' k)^x \ldots$$

5. In the same manner, if the system of equations to be solved had been

$$\left.\begin{array}{l} \Delta^n . u_x = a_1 u_x + b_1 v_x + c_1 w_x + \ldots \\ \Delta^n . v_x = a_2 u_x + b_2 v_x + c_2 w_x + \ldots \\ \Delta^n . w_x = a_3 u_x + b_3 v_x + c_3 w_x + \ldots \\ \qquad\qquad \text{\&c.} \end{array}\right\},$$

we should have for the reduct equation

$$\Delta^n . (\lambda u_x + \mu v_x + \nu w_x + \ldots) = k^n (\lambda u_x + \mu v_x + \nu w_x + \ldots),$$

the solution of which is, α, α', α'', &c., being, as before, the n several roots of unity,

$$\lambda u_x + \mu v_x + \nu w_x + \ldots = C(\alpha k + 1)^x + C'(\alpha' k + 1)^x + C''(\alpha'' k + 1)^x + \ldots$$

EXAMPLES.

(1.) Let it be proposed to solve the system of the second order

$$\left.\begin{array}{l} u_{x+2} = b_1 v_x + c_1 w_x \\ v_{x+2} = a_2 u_x + c_2 w_x \\ w_{x+2} = a_3 u_x + b_3 v_x \end{array}\right\}.$$

The equations of condition in this case are

$$\left.\begin{array}{l} a_2 \mu + a_3 \nu = k^2 \lambda \\ b_1 \lambda + b_3 \nu = k^2 \mu \\ c_1 \lambda + c_2 \mu = k^2 \nu \end{array}\right\};$$

the reduct equation

$$e^{2D} . (\lambda u_x + \mu v_x + \nu w_x) = k^2 (\lambda u_x + \mu v_x + \nu w_x);$$

the solution of this equation,

$$\lambda u_x + \mu v_x + \nu w_x = C k^x + C'(-k)^x;$$

while the equation to determine k is

$$k^6 - (b_1 a_2 + c_1 a_3 + c_2 b_3) k^2 - (c_2 b_1 a_3 + c_1 a_2 b_3) = 0.$$

(2.) If it be proposed to solve the system

$$\left.\begin{array}{l} \Delta^2 . u_x = b_1 v_x + c_1 w_x \\ \Delta^2 . v_x = a_2 u_x + c_2 w_x \\ \Delta_2 . w_x = a_3 u_x + b_3 v_x \end{array}\right\},$$

we have, for the reduct equation,

$$\Delta^2 . (\lambda u_x + \mu v_x + \nu w_x) = k^2 (\lambda u_x + \mu v_x + \nu w_x);$$

for the solution of this equation

$$\lambda u_x + \mu v_x + \nu w_x = C(1+k)^x + C'(1-k)^x;$$

while k is determined by the same equation as in the last example.

As this equation is of the sixth degree, it might be supposed that the complete solution of the problem should consist of six equations, each involving two arbitrary constants. It will be observed, however, that since the roots of the equation in k are of the form

$$\pm k_1, \qquad \pm k_2, \qquad \pm k_3,$$

and λ, μ, ν depend only on k^2, these six equations, each of which is of the shape, in the former case

$$\lambda u_x + \mu v_x + \nu w_x = Ck^x + C'(-k)^x,$$

and in the latter case,

$$\lambda u_x + \mu v_x + \nu w_x = C(1+k)^x + C'(1-k)^x,$$

are reducible to three, and there are not virtually more than six arbitrary constants. These constants are, in general, to be determined by given values of u_x, v_x, w_x, u_{x+1}, v_{x+1}, w_{x+1}, corresponding to a given value of x.

6. If the system of simultaneous equations, proposed for solution, were given in the form,

$$\left.\begin{array}{c} a_1 u_{x+n} + b_1 v_{x+n} + c_1 w_{x+n} + \ldots = u_x \\ a_2 u_{x+n} + b_2 v_{x+n} + c_2 w_{x+n} + \ldots = v_x \\ a_3 u_{x+n} + b_3 v_{x+n} + c_3 w_{x+n} + \ldots = w_x \\ \text{\&c.} \end{array}\right\},$$

or, in the correlative form,

$$\left.\begin{array}{c} a_1 \Delta^n u_x + b_1 \Delta^n v_x + c_1 \Delta^n w_x + \ldots = u_x \\ a_2 \Delta^n u_x + b_2 \Delta^n v_x + c_2 \Delta^n w_x + \ldots = v_x \\ a_3 \Delta^n u_x + b_3 \Delta^n v_x + c_3 \Delta^n w_x + \ldots = w_x \\ \text{\&c.} \end{array}\right\},$$

the first equation being multiplied by λ, the second by μ, the third by ν, &c., and all being added together, subject to the conditions

$$\left.\begin{aligned} a_1\lambda + a_2\mu + a_3\nu + \ldots &= \frac{\lambda}{k^n} \\ b_1\lambda + b_2\mu + b_3\nu + \ldots &= \frac{\mu}{k^n} \\ c_1\lambda + c_2\mu + c_3\nu + \ldots &= k^n \\ \&\text{c.} & \end{aligned}\right\},$$

we get for the reduct equation, in the former case,

$$e^{nD}.(\lambda u_x + \mu v_x + \nu w_x + \ldots) = k^n(\lambda u_x + \mu v_x + \nu w_x + \ldots),$$

and, in the latter case,

$$\Delta^n.(\lambda u_x + \mu v_x + \nu w_x + \ldots) = k^n(\lambda u_x + \mu v_x + \nu w_x + \ldots);$$

the corresponding solutions being, in the former case,

$$\lambda u_x + \mu v_x + \nu w_x + \ldots = C(\alpha k)^x + C'(\alpha' k)^x + C''(\alpha'' k)^x + \ldots,$$

and, in the latter case,

$$\lambda u_x + \mu v_x + \nu w_x + \ldots = C(\alpha k + 1)^x + C'(\alpha' k + 1)^x + C''(\alpha'' k + 1)^x + \ldots$$

It should be noticed that the values of the constants and of the several roots of the equation in k are, of course, wholly different from those occurring in the previous article, in which a notation similar to that just used was employed.

7. If the system of equations to be solved were of the form

$$\left.\begin{aligned} \Delta.u_x &= a_1u_x + b_1v_x + c_1w_x + \ldots + f_1(x) \\ \Delta.v_x &= a_2u_x + b_2v_x + c_2w_x + \ldots + f_2(x) \\ \Delta.w_x &= a_3u_x + b_3v_x + c_3w_x + \ldots + f_3(x) \\ &\&\text{c.} \end{aligned}\right\},$$

where f_1, f_2, f_3, &c., are given algebraic functions,—proceeding as before, and with the same system of conditions, we obtain the equation

$$\Delta(\lambda u_x + \mu v_x + \&\text{c.}) = k(\lambda u_x + \mu v_x + \&\text{c.}) + (\lambda f_1 + \mu f_2 + \&\text{c.}),$$

or

$$(\Delta - k).(\lambda u_x + \mu v_x + \nu w_x + \&\text{c.}) = \lambda f_1 + \mu f_2 + \nu f_3 + \&\text{c.} = F(x),$$

the solution of which is, in its primary symbolic form,

$$\lambda u_x + \mu v_x + \nu w_x + \&\text{c.} = (\Delta - k)^{-1}.F(x) + (\Delta - k)^{-1}.0,$$

or, in its semi-evaluated form, supposing, for simplicity, that F only contains positive integer values of x,

$$\lambda u_x + \mu v_x + \nu w_x + \&c. = -\frac{1}{k}\left(1 + \frac{\Delta}{k} + \frac{\Delta^2}{k^2} + \&c.\right). F(x) + Ck^x.$$

The operations indicated by the symbols Δ, Δ^2, &c., being performed, the complete solution, in its primary type, is obtained, it being observed that λ, μ, ν, &c., enter linearly in the right-hand member.

A corresponding method of solution, of course, will apply to such a system of equations as

$$\left.\begin{array}{l} u_{x+1} = a_1 u_x + b_1 v_x + c_1 w_x + \ldots + f_1(x) \\ v_{x+1} = a_2 u_x + b_2 v_x + c_2 w_x + \ldots + f_2(x) \\ w_{x+1} = a_3 u_x + b_3 v_x + c_3 w_x + \ldots + f_3(x) \\ \qquad\qquad \&c. \end{array}\right\},$$

or we may, in some cases with advantage, employ an extension of the method stated in the second article.

8. If the system of equations proposed for solution were of the form

$$\left.\begin{array}{l} \Phi(\Delta) . u_x + \Psi(\Delta) . v_x = F_1(x) \\ \Phi(\Delta) . v_x - \Psi(\Delta) . u_x = F_2(x) \end{array}\right\},$$

where F_1 and F_2 are given functions of x, we may proceed in the following manner.

Operating upon the first equation with $\Phi(\Delta)$, and making substitution from the second equation, we get

$$\Phi(\Delta)^2 . u_x + \Psi(\Delta)^2 . u_x = \Phi(\Delta) . F_1(x) - \Psi(\Delta) . F_2(x).$$

The operations susceptible of execution being performed, this equation is obviously reducible to the form

$$\{\Phi(\Delta)^2 . + \Psi(\Delta)^2 .\} u_x = F_3(x),$$

in which there is now but a single unknown function.

This last equation, in general, admits of solution, and the value of u_x being found, that of v_x is obtained by substitution in either of the given equations.

A mode of solution precisely similar will apply to the system correlative to the above, namely,

$$\left.\begin{array}{l} (a_0 u_x + a_1 u_{x+1} + a_2 u_{x+2} + \ldots + a_n u_{x+n}) + (b_0 v_x + b_1 v_{x+1} + b_2 v_{x+2} + \ldots + b_m v_{x+m}) = F_1(x) \\ (a_0 v_x + a_1 v_{x+1} + a_2 v_{x+2} + \ldots + a_n v_{x+n}) - (b_0 u_x + b_1 u_{x+1} + b_2 u_{x+2} + \ldots + b_m u_{x+m}) = F_2(x) \end{array}\right\}$$

or

$$\left.\begin{array}{l} \Phi(e^D) . u_x + \Psi(e^D) . v_x = F_1(x) \\ \Phi(e^D) . v_x - \Psi(e^D) . u_x = F_2(x) \end{array}\right\},$$

the equations being written in their condensed symbolic form.

EXAMPLES.

(1.) Let the system proposed for solution be

$$\left.\begin{array}{l}(a_0u_x + a_2u_{x+2} + a_4u_{x+4}) + (a_1v_{x+1} + a_3v_{x+3}) = m\alpha^x \\ (a_0v_x + a_2v_{x+2} + a_4v_{x+4}) - (a_1u_{x+1} + a_3u_{x+3}) = n\beta_x\end{array}\right\}.$$

The reduct equation is in this case

$$(a_0 + a_2e^{2D} + a_4e^{4D})^2 . u_x + (a_1e^D + a_3e^{3D})^2 . u_x = m(a_0 + a_2\alpha^2 + a_4\alpha^4)\alpha^x \\ - n(a_1\beta + a_3\beta^3)\beta^x,$$

which, as is readily seen, may be written in the shape

$$F(e^{2D}) . u_x = A\alpha^x - B\beta^x,$$

where F is a biquadratic of given form, in which the coefficient of the highest term has been reduced to unity.

The solution of this equation is, omitting the arbitrary portion,

$$u_x = \frac{A}{F(\alpha^2)} . \alpha^x - \frac{B}{F(\beta^2)} . \beta^x,$$

and the arbitrary portion of the solution, itself, is

$$\begin{array}{c}C_1k_1^x + C'_1(-k_1)^x + C_2k_2^x + C'_2(-k_2)^x \\ + \\ C_3k_3^x + C'_3(-k_3)^x + C_4k_4^x + C'_4(-k_4)^x,\end{array}$$

if the roots of the biquadratic $F(k^2) = 0$ be supposed to be

$$k_1^2, \quad k_2^2, \quad k_3^2, \quad k_4^2.$$

The value of u_x being thus found, the value of v_x is had by simple substitution in either of the given equations, and the mode of determination of the arbitrary constants may be easily deduced from the previous articles.

(2.) If the system of equations proposed for solution were

$$\left.\begin{array}{l}(a_0u_x + a_2\Delta^2u_x + a_4\Delta^4u_x) + (a_1\Delta v_x + a_3\Delta^3v_x) = m\alpha^x \\ (a_0v_x + a_2\Delta^2v_x + a_4\Delta^4v_x) - (a_1\Delta u_x + a_3\Delta^3u_x) = n\beta^x\end{array}\right\},$$

we should have for the reduct equation

$$F(\Delta^2)\, u_x = A'\alpha^x - B'\beta^x,$$

the solution of which is, omitting as before the arbitrary portion,

$$u_x = \frac{A'}{F\{(\alpha-1)^2\}}\alpha^x - \frac{B'}{F\{(\beta-1)^2\}}\beta^x,$$

and the arbitrary portion is

$$\begin{gathered}C_1(1+k_1)^x + C'_1(1-k_1)^x + C_2(1+k_2)^x + C'_2(1-k_2)^x \\ + \\ C_3(1+k_3)^x + C'_3(1-k_3)^x + C_4(1+k_4)^x + C'_4(1-k_4)^x.\end{gathered}$$

9. Let the system proposed for solution be

$$\left.\begin{aligned}u_{x+2} &= a_1u_x + b_1v_x + c_1 \\ v_{x+2} &= a_2u_x + b_2v_x + c_2\end{aligned}\right\}.$$

Proceeding as above, we obtain

$$e^{2D}.(\lambda u_x + \mu v_x) = k^2(\lambda u_x + \mu v_x) + \lambda c_1 + \mu c_2,$$

whence, at once, result the two equations for the determination of u_x and v_x, namely,

$$\left.\begin{aligned}\lambda u_x + \mu v_x &= C_1k_1^x + C'_1(-k_1)^x + \frac{\lambda c_1 + \mu c_2}{1-k_1^2} \\ \lambda' u_x + \mu' v_x &= C_2k_2^x + C'_2(-k_2)^x + \frac{\lambda' c_1 + \mu' c_2}{1-k_2^2}\end{aligned}\right\}$$

the equation in k being, as before,

$$(k^2 - a_1)(k^2 - b_2) = b_1a_2.$$

Hence,

$$u_x = \frac{c_1(1-b_2) + b_1c_2}{(1-a_1)(1-b_2) - b_1a_2} + D_1k_1^x + D'_1(-k_1)^x + D_2k_2^x + D'_2(-k_2)^2,$$

$$v_x = \frac{c_2(1-a_1) + a_2c_1}{1-a_1.1-b_1 - b_1a_2} + \frac{k_1^2 - a_1}{b_1}\{D_1k_1^x + D'_1(-k_1)^x\}$$
$$+ \frac{k_2^2 - a_1}{b_1}\{D_2k_2^x + D'_2(-k_2)^x\}.$$

10. As the values of the constants a_1, b_1, c_1, &c., are supposed to be given by observation, and are therefore liable to certain small errors, it may be worth while to consider what corrections should be introduced, if these constants should become, respectively, $a_1 + \delta a_1$, $b_1 + \delta b_1$, $c_1 + \delta c_1$, &c.

It is obvious that, as the roots of the equation in k depend on the values of the constants stated, these roots will receive certain increments, which may be, for convenience, respectively denominated by the expressions δk_1, δk_2, δk_3, &c.

Thus in the second example, quoted under the first article, the value of u_x will become

$$u_x = C_1(k_1 + \delta k_1)^x + C_2(k_2 + \delta k_2)^x + C_3(k_3 + \delta k_3)^x,$$

or, since δk_1, δk_2, δk_3, are very small quantities,

$$u_x = C_1 (k_1^x + x k_1^{x-1} \delta k_1) + C_2 (k_2 + x k_2^{x-1} \delta k_2) + C_3 (k_3 + x k_3^{x-1} \delta k_3),$$

or,

$$u_x = (C_1 k_1^x + C_2 k_2^x + C_3 k_3^x) + x (C_1 k_1^{x-1} \delta k_1 + C_2 k_2^{x-1} \delta k_2 + C_3 k_3^{x-1} \delta k_3).$$

There is no difficulty in determining the corresponding corrections to be made on the values of v_x and w_x.

As regards the method exhibited in this section, it may be allowed to state that it is precisely similar to that employed for the solution of systems of simultaneous differential equations, and, as I believe, for the first time published in a treatise on the "Calculus of Operations," in the year 1855.

Postscript.—Upon communicating some of the results contained in this section to the Rev. Dr. Lloyd, with a view to the suggestion by him of some physical applications, if such existed, I received from him a statement of his views, which it is right to lay before the Academy. He points out how, in the present position of physics, the conditions of the problems discussed are, in general, first treated with respect to some one predominant element, and then the other elements taken up and dealt with as residual phenomena, or disturbing causes. Dr. Lloyd states his belief that any attempt to apply a more rigorous method (in which all the elements are simultaneously taken into account) would fail, in consequence of the large errors, which the errors of the observed results would entail, in the process of elimination. Professor Haughton has expressed his coincidence in this view. As a matter of course, I defer to the opinions expressed by physicists so justly distinguished as these gentlemen, and put forward the methods contained in this paper simply for their mathematical value, whatever that may be. One remark only I would venture to offer. Is it not possible that we may yet be enabled to state, by such systems of equations, the conditions of phenomena which depend on the simultaneous action of heterogeneous laws: for instance, those of heat, electric action, and chemical affinity, supposing for a moment the division of these agencies to be logically just? The geometrical illustration proposed in the first article would possibly give some reason to hope that this calculus of finite differences may yet be made more ancillary to physical research than it hitherto has been.

The Secretary read the following letter from the Right Hon. Edward Cardwell, addressed to the President:—

"*Dublin Castle, 22nd October*, 1859.

"Sir,—In reference to your letter of the 17th ultimo, I am directed by the Lord Lieutenant to acquaint you, that the Lords Commissioners of Her Majesty's Treasury have been pleased to sanction a grant of two hundred pounds in aid of the completion of a Catalogue of the Museum of the Royal Irish Academy.

"Their Lordships have given the necessary authority to the Paymaster of Civil Services for the issue of the above-mentioned sum.

"I am, Sir, your obedient servant,

(Signed) "EDWARD CARDWELL.

"*Rev. James Henthorn Todd, D. D.*,
"*President of the Royal Irish Academy.*"

Moved by the Rev. Charles Graves, D. D., seconded by the Rev. J. Carson, D. D., and—

RESOLVED,—That the President of the Academy be requested to convey to His Excellency the Lord Lieutenant the marked thanks of the Academy, for having thus exerted his influence in its favour with the Lords Commissioners of Her Majesty's Treasury.